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NORTHWESTERN UNIVERSITY

EVANSTON, ILLINOIS

SYSTEMS RESEARCH MEMORANDUM No. 128

The Technological Institute

The College of Arts and Sciences

Northwestern University

A GENERALIZATION OF A THEOREM OF EDMUND EISENBERG

by

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August 1965

This paper is an extension and revision of ONR Research Report No. 92.

Part of the research underlying this report was undertaken for the Office of Naval Research, Contract Nonr-1228(10), Project NR 047-021, for the U. S. Public Health Service, Divisions of Water Supply and Pollution Control and Environmental Engineering and Food Protection, Contracts WP 00019-06 and EF 00355-03, and for the U. S. Army Research Office - Durham, Contract No. DA-31-124-ARO-D-322, at Northwestern University.

SYSTEMS RESEARCH GROUP

A. Charnes, Director

The convex duality theory as developed by Charnes-Cooper-Kortanek which is valid for non-differentiable quasi-concave functions is applied to derive a gradient inequality for a non-differentiable function involving a symmetric quadratic form with positivity required only on a given convex polyhedral cone, generalizing a theorem of Edmund Eisenberg. No assumption is needed either that the primal problem attain its extremal value, since the CCK duality theory includes this possibility. The theory directly applies to the non-homogeneous problem where now the given cone is replaced by the intersection of a finite number of half spaces.

If $f(x)$ is a convex differentiable function over R^n and A is a real $m \times n$ matrix, then the gradient inequality $(x-x_o)^T f_{x_o} \geq 0$ whenever $Ax \leq 0$ holds if and only if x_o minimizes $f(x)$ over $X = \{x | Ax \leq 0\}$. This inequality has a Farkas-Minkowski equivalent which follows directly from the Farkas-Minkowski lemma. This dual equivalent is exactly the statement of convex duality as developed by Charnes-Cooper-Kortanek 1/ which is also valid for non-differentiable convex functions. It is this statement of convex duality that E. Eisenberg 2/ and S. M. Sinha 3/ developed in an elegant manner for the important special case, $f(x) = a^T x + (x^T Cx)^{\frac{1}{2}}$.

1/ See [5], pp. 605-608.

2/ See [10].

3/ See [9]. This development parallels the methods of Eisenberg's earlier article, "Supports of a Convex Function," Bulletin of the AMS, Vol. 68, No. 3, May 1962, pp. 192-195, to prove a dual theorem for a quadratic

$$\phi(x) = a^T x + \sum_{i=1}^{t} (x^T C_i x)^{\frac{1}{2}} ,$$

where C_i are as above.

Part of their concern (e.g., Sinha) has been that one cannot directly apply the usual convex programming methods to the problem,

$$\min\ a^T x + (x^T Cx)^{\frac{1}{2}} \quad \text{with } Ax \le b\ , \ \text{ and } x \ge 0$$

because of non-differentiability of $(x^T Cx)^{\frac{1}{2}}$. However, this type of non-differentiability can be overcome, as well as that of the more general non-differentiable constraints $A^i x + (x^T B^i x)^{\frac{1}{2}} \le b$, by a simple transformation and the introduction of "spacing" variables. This device of Charnes and Cooper may be found, for example, in Charnes-Cooper's "Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints," Operations Research, Vol. 11, No. 1, pp. 18-39, January-February 1963, (which was circulated as an ONR report in 1960-61) or in the Charnes-Cooper-Thompson paper, "Characterizations by Chance-Constrained Programming" at the Symposium of Mathematical Programming, June 1962, whose proceedings were published as Recent Advances in Mathematical Programming, R. L. Graves and P. Wolfe, editors.

In particular the extended dual equivalent of the "gradient" inequality for the non-differentiable symmetric function, $f(x)$, of Eisenberg will now be directly derived from the CCK duality theory, in a manner which generalizes Eisenberg's theorem. 1/

Theorem Assume that $f(x) = a^T x + (x^T Cx)^{\frac{1}{2}} \ge 0$ whenever $x \in X = \{x | Ax \le 0\}$, where C is symmetric and $x^T Cx \ge 0$ on X. Then there exist $\pi \ge 0$ and z such that

(i) $Az \le 0$

(ii) $\pi^T A + a^T + z^T C = 0$ and $z^T Cz \le 1$.

1/ See E. Eisenberg, [10], Theorem 3; our theorem, however, is a generalization since we do not require $x^T Cx \ge (a^T x)^2$.

Proof Consider the problem:

(I)

$$\begin{array}{rl} \min & a^T x + t \\ \text{subject to} & -Ax \geq 0 \\ & -x^T Cx + t^2 \geq 0 \\ & t \geq 0 \; . \end{array}$$

Assume for the moment that the constraint set is convex, which would be the case, for example, if $-x^T Cx + t^2$ were quasi-concave. It should be emphasized that in convex duality theory, concave constraint functions are not needed per se, but only that the constraint set which they determine be convex. Generalizations to quasi-concave constraint functions were developed by Arrow-Hurwicz-Uzawa and by Arrow-Enthoven. 1/

In our formulation the set determined by $-x^T Cx + t^2 \geq 0$ alone is not convex. However, the additional inequality $t \geq 0$ chooses one nappe of the elliptic hyperboloid determined by the quadratic inequality, thus defining a convex set whose differentiable support system is now determined by usual processes of differentiation. Observe that the infimum exists, and in fact a minimum value of zero is actually attained. Introducing this support system brings problem (I) into the following equivalent inequality form.

(I)

$$\begin{array}{rll} \min & a^T x + t & \\ \text{subject to} & -Ax \geq 0 & \\ & x^T(-Cx_\alpha) + t \geq 0 & \text{for all } x_\alpha \text{ with } x_\alpha^T Cx_\alpha = 1,\ -Ax_\alpha \geq 0 \quad 2/ \\ & t \geq 0 & \end{array}$$

1/ See [1] and [2].

2/ Since the cone $a^T x + (x^T Cx)^{\frac{1}{2}} \geq 0$ contains $-Ax \geq 0$ by hypothesis, one may as we have done, further restrict x_α to range over $-Ax_\alpha \geq 0$. If $x^T Cx = 0$ on $Ax \leq 0$, then the ordinary Farkas-Minkowski lemma applies.

Lemma 1 The linear inequality system for (I) as presented above completely determines the convex constraint set, $-Ax \geq 0$, $-x^T Cx + t^2 \geq 0$, $t \geq 0$.

Proof Over $x \in X$ and $t \geq 0$, the inequality $-x^T Cx + t^2 \geq 0$ determines a convex set, and therefore for any (x_α, t_α) with $x_\alpha^T Cx_\alpha = t_\alpha^2$ and non-vanishing gradient, $(-2Cx_\alpha, 2t_\alpha) \neq 0$, it follows that

$$x^T(-Cx_\alpha) + tt_\alpha \geq 0 \; .$$ 1/

Therefore, if (x, t) is feasible for problem (I), it satisfies the system

$$x^T(-Cx_\alpha) + t \geq 0 \text{ for all } x_\alpha \text{ with } x_\alpha^T Cx_\alpha = 1$$
$$t \geq 0 \; .$$

Conversely, if (x, t) satisfies this system and $-Ax \geq 0$, then $-x^T Cx + t^2 \geq 0$; for in the trivial case of $x^T Cx = 0$, we are done. In the case of $x^T Cx \neq 0$, set $x_\alpha = \dfrac{x}{(x^T Cx)^{\frac{1}{2}}}$ so that $x_\alpha^T Cx_\alpha = 1$ and $-Ax_\alpha \geq 0$, and therefore x_α is a member of our indexing set. Therefore,

$$x^T\left(\frac{-Cx}{(x^T Cx)^{\frac{1}{2}}}\right) + t \geq 0 \Longrightarrow t \geq (x^T Cx)^{\frac{1}{2}} \Longrightarrow t^2 \geq x^T Cx \; . \qquad \text{Q.E.D.}$$

Lemma 2 The linear inequality system for (I) is a Farkas-Minkowski system.

Proof Since problem (I) has finite optima it suffices to consider bounded coefficients because it is only these which the differential system need encompass, i.e., it suffices to consider a compact index set, I, contained in $\{x_\alpha \mid x_\alpha^T Cx_\alpha = 1, \; Ax_\alpha \leq 0\}$. Therefore the possibly infinite differential

---

1/ Compare [2], p. 788, where the property of quasi-concavity of the constraint functions, g^j, is used to establish this fact. They point out that for this purpose it suffices that the constraint set be convex.

subsystem has interior points. We now distinguish two cases on the remaining finite subsystem, $-Ax \geq 0$.

Case 1 There exists x_o such that $-Ax_o > 0$. It then follows that (x_o, t_o) is an interior point for the entire system for t_o sufficiently large. Thus, we have a canonically closed Haar system and therefore a Farkas-Minkowski system.

Case 2 The system $Ax \leq 0$ has no interior points, i.e., $X = \{x | Ax = 0\}$ and X is an $(n-r)$ flat, $1 \leq r \leq m$. In this case we may choose r linearly independent equations from this subsystem which also determine X , say $A_1x_1 + A_2x_2 = 0$, where A_1 is $r \times r$ and nonsingular and A_2 is $r \times (n-r)$. Therefore, problem (I) becomes

$$
\begin{array}{lll}
\min & a_1^T x_1 + a_2^T x_2 & + t \\
\text{subject to} & A_1 x_1 + A_2 x_2 & = 0 \\
& x_1^T(-C_1 x_\alpha) + x_2^T(-C_2 x_\alpha) + t \geq 0 \quad , & x_\alpha \in I \\
& t \geq 0 &
\end{array}
$$

where C_1 is $r \times n$, C_2 if $(n-r) \times n$ and $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C$.

To prove the Farkas-Minkowski property for the given system we first reduce variables via $x_1 = -A_1^{-1}A_2x_2$,[1/] to attain:

$$(I_r)$$

$$
\begin{array}{ll}
\min & (a_2^T - a_1^T A_1^{-1} A_2) x_2 \qquad + t \\
\text{subject to} & x_2^T(-C_2 x_\alpha + A_2^T (A^{-1})^T C_1 x_\alpha) + t \geq 0 , \quad x_\alpha \in I \\
& t \geq 0
\end{array}
$$

1/ See [12] for a general discussion and proof of a similar technique for resolving duality gaps for infinite linear inequality systems which do not have interior points.

Since the original system is compact, the reduced system also is and, in addition, contains no trivial inequalities.

Clearly the linear inequality system of (I_r) has interior points. Therefore, it is a Haar system and the semi-infinite duality theorem applies to (I_r) and its dual (II_r):

$$(II_r)$$

$$\begin{array}{lll} \max & \sum_\alpha 0\cdot\lambda_\alpha & +0\mu \\ \text{subject to} & \sum_\alpha(-x_\alpha^T C_2^T + x_\alpha^T C_1^T A_1^{-1} A_2)\lambda_\alpha & = a_2^T - a_1^T A_1^{-1} A_2 \\ & \sum_\alpha \lambda_\alpha & +\mu = 1 \\ & & \lambda_\alpha, \mu \geq 0 \end{array}$$

Therefore, let $(x_2^*, t^*; \lambda^*, \mu^*)$ be a dual optimal solution. In order to go from the reduced system (I_r) to the original system (I), simply set $x_1^{*T} = -x_2^{*T} A_2^T (A_1^{-1})^T$ and $\omega^{*T} = a_1^T A_1^{-1} + \sum_\alpha x_\alpha^T C_1^T A_1^{-1} \lambda_\alpha^*$. Therefore $(x_1^*, x_2^*, t^*; \omega^*, \lambda^*, \mu^*)$ is a dual optimal solution for (I) as we now easily check. First, the dual problem to (I) is of course:

$$(II)$$

$$\begin{array}{llll} \max & \omega^T\cdot 0 + \sum_\alpha 0\cdot\lambda_\alpha & +0\cdot\mu & \\ \text{subject to} & \omega^T A_1 + \sum_\alpha(-x_\alpha^T C_1^T)\lambda_\alpha & & = a_1^T \\ & \omega^T A_2 + \sum_\alpha(-x_\alpha^T C_2^T)\lambda_\alpha & & = a_2^T \\ & \sum_\alpha \lambda_\alpha & +\mu & = 1 \\ & \lambda, \mu \geq 0\,. & & \end{array}$$

To check dual feasibility for (II), observe first that the expression for a_1^T is verified directly by substitution for ω^{*T} and λ^*. The required expression for a_2^T is obtained by substituting for ω^{*T} into the first constraint of (II_r), i.e.,

$$a_2^T = a_1^T A_1^{-1} A_2 + \sum_\alpha (-x_\alpha^T C_2^T + x_\alpha^T C_1^T A_1^{-1} A_2) \lambda_\alpha^* = \omega^{*T} A_2 - \sum_\alpha x_\alpha^T C_2^T \lambda_\alpha^* \ .$$

Finally the equality of the (I) - (II) functionals follows directly from the equality of the (I_r)-(II_r) functionals. Therefore the proof of Lemma 2 is complete. Thus, because of Lemma 2, there is no loss of generality if we assume that $Ax \leq 0$ has interior points; a Farkas-Minkowski system is always attained, and therefore the semi-infinite duality theory applies to the dual problems (I) and (II). For future reference in the proof of our theorem, we rewrite problem (II) in its original form.

(II)

$$\begin{aligned}
&\max \quad \omega^T \cdot 0 + \sum_\alpha 0 \cdot \lambda_\alpha + \mu \cdot 0 \\
&\text{subject to } -\omega^T A + \sum_\alpha (-x_\alpha^T C) \lambda_\alpha = a^T \\
&\qquad \sum_\alpha \lambda_\alpha + \mu = 1
\end{aligned}$$

where $\omega^T, \lambda^T, \mu \geq 0$ and λ^T have only finitely many non-zero components.

Now the minimum of problem (I) is assumed at $x^* = 0$, $t^* = 0$, and possibly at other points with $t^* \neq 0$. Assume, in general, that (x^*, t^*) is an optimal solution to (I). A dual optimal solution exists by the extended dual theorem, say, $(\omega^{T*}, \lambda^{T*}, \mu^*)$. Now since $0 \epsilon X$ and $\sum_\alpha \lambda_\alpha^* + \mu^* = 1$, if follows that

$$z = \sum_\alpha x_\alpha^* \lambda_\alpha^* + 0 \cdot \mu^* \epsilon X \quad \text{(i.e., feasible) since all}$$

x_α^*'s are feasible.

Therefore, $(-\Sigma x^*_\alpha \lambda^*_\alpha)^T C x_\beta + t^* \geq 0$ for all x_β with $x_\beta^T C x_\beta = 1$, $-Ax_\beta \geq 0$. In particular, $\lambda^*_\alpha t^* \geq \lambda^*_\alpha (\Sigma x^*_\alpha \lambda^*_\alpha)^T Cx^*_\alpha$, and summing we get

$$t^* \geq \Sigma \lambda^*_\alpha t^* \geq (\Sigma x^*_\alpha \lambda^*_\alpha)^T C(\Sigma x^*_\alpha \lambda^*_\alpha) \quad , \text{ i.e.,}$$

$$z^T Cz \leq t^*$$

This inequality is with respect to any optimal solution to problem (I), (x^*, t^*). Any non-degenerate optimal solution, i.e., $x^{*T} Cx^* \neq 0$ yields an equivalent one with $t^* = 1$, namely, $a^T \frac{x^*}{(x^{*T} Cx^*)^{\frac{1}{2}}} + 1 = 0$. Thus, the inequality becomes $z^T C z \leq 1$, which of course already includes the case of degenerate solutions.

Therefore, the z that occurs in the optimal dual solution is exactly the one required of the theorem, i.e.,

$$z^T C z \leq 1$$

$$a^T + z^T C + \omega^T A = 0 \quad \text{and} \quad Az \leq 0 .$$

To complete the proof of the theorem we consider the general case where the constraint set $Ax \leq 0$, $-x^T Cx + t^2 \geq 0$, $t \geq 0$ may not be convex if all that we require of C is that it be symmetric and $x^T Cx \geq 0$ on $Ax \leq 0$. For example, the set given by: $t^2 - x_1 x_2 \geq 0$, $t \geq 0$, and $x_1, x_2 \geq 0$ is not convex. However, we may still apply our duality theory.

Consider the <u>partial</u> support system given by

$$\begin{aligned} -Ax &\geq 0 \\ x^T(-Cx_\alpha) + t &\geq 0 \quad \text{for } x_\alpha \in I , \\ t &\geq 0 \end{aligned}$$

where I is defined as follows:

$$I = \{x_\alpha | x_\alpha^T C x_\alpha = 1 \ , \ \ Ax_\alpha \leq 0\} \cap \{x | \|x\| \leq k\}$$

for some fixed (large) $k > 0$. Thus, I is a compact set in R_m. Since $x^T Cx$ is continuous, it is bounded on I, and the image is compact. Further, the infinite subsystem has interior points, as seen by simply taking some $t > \max_{\alpha \in I} |x_\alpha^T Cx_\alpha|$. Therefore by Lemma 2 we again obtain a Farkas-Minkowski system.

To show that this minimization problem is bounded below, we need only show that the dual problem is feasible. Following Eisenberg,[1/] we now show that the equality system,

$$\pi^T(-A) + z^T(-C) = a^T$$

$$Az + y = 0 \ ; \ \pi, y \geq 0$$

is consistent. By the finite Farkas-Minkowski theorem this is equivalent to showing that $a^T u \geq 0$ whenever

$$\begin{aligned} -Au \quad\quad &\geq 0 \\ -Cu + A^T v &= 0 \\ v &\geq 0 \ . \end{aligned}$$

To this end, assume (u, v) satisfies the above system. Claim $u^T Cu \leq 0$; for $u^T Cu = u^T A^T v = v^T Au$, and $v^T Au \leq 0$ since $Au \leq 0$ and $v \geq 0$. On the other hand, $Au \leq 0 \Rightarrow u^T Cu \geq 0$; hence $u^T Cu = 0$. Thus $Au \leq 0 \Rightarrow a^T u + (u^T Cu)^{\frac{1}{2}} \geq 0$, which finally implies $a^T u \geq 0$. Observe that we have used all the assumptions of the theorem.

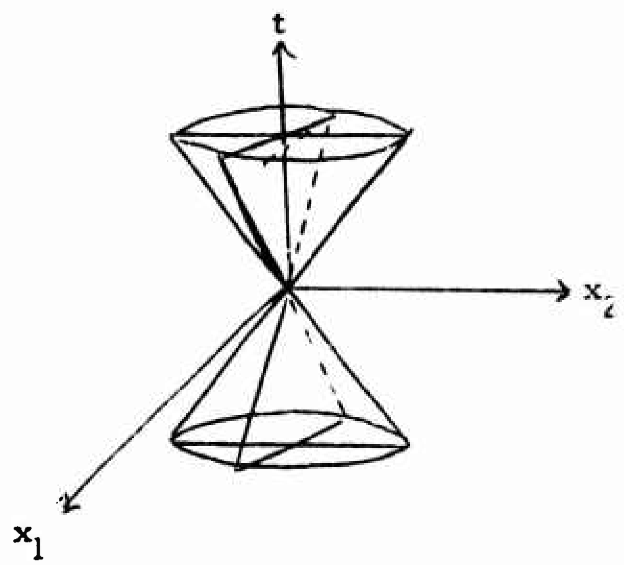
1/ See E. Eisenberg [10], Appendix B, where a similar argument is presented for dual feasibility.

Thus, the dual problem is feasible, and by taking k large enough we can include enough points in the index set I so that the partial support system contains the proper coefficients which permit dual feasibility. This completes the proof of the theorem as stated.

In closing, we remark that the general convex duality theory also provides analogous results for the related non-homogeneous problem 1/ where now the feasibility region is given by $Ax \leq b$. In this general context, the minimum need not be assumed, which is consistent with the Extended Dual Theorem. However, to avoid this situation in the statement of duality, Sinha 2/ introduces an assumption which requires the existence of a feasible solution (π, z) with $z^T Cz < 1$ if the dual problem is feasible. This assumption is not necessary for duality theory because the extended dual theorem includes the possibility of the primal problem <u>not</u> attaining its extremal value. 3/ To illustrate this phenomenon consider the following example introduced by Sinha 4/ and its semi-infinite equivalent.

(I)

$$\begin{array}{ll} \min & -x_2 + t \\ \text{subject to} & -x_1^2 - x_2^2 + t^2 \geq 0 \\ & x_1 \geq 2 \\ & t \geq 0 . \end{array}$$



1/ See [10], section III.

2/ See [9]. p. 15, for an exact statement of this assumption.

3/ For an example of this situation see [8], p. 215.

4/ See Sinha, ibid., p. 15.

In this example, $\inf(-x_2 + t) = 0$; for clearly $t - x_2 \geq 0$ over the constraint set, and taking $x_1 = 2$, $x_2 = n$, and $t = n + 2/n$ for $n = 1, 2, \ldots,$ proves the assertion. Note that this infimum is not assumed. Introducing supports, (I) becomes:

(I)

$$
\begin{array}{lll}
\min & -x_2 + t & \\
\text{subject to} & x_1 \geq 2 & \\
 & x_1(-x_1^\alpha) + x_2(-x_2^\alpha) + t \geq 0 & \text{for all } (x_1^\alpha, x_2^\alpha) \text{ with } (x_1^\alpha)^2 + (x_2^\alpha)^2 = 1.^{1/} \\
 & t \geq 0 &
\end{array}
$$

The dual is:

(II)

$$
\begin{array}{lll}
\max & 2\omega & \\
\text{subject to} & \omega + \sum_\alpha (-x_1^\alpha)\lambda_\alpha & = 0 \\
 & \sum_\alpha (-x_2^\alpha)\lambda_\alpha & = -1 \\
 & \sum_\alpha \lambda_\alpha \quad + \mu & = 1
\end{array}
$$

where $\omega, \lambda, \mu \geq 0$.

Now, taking $x_1^{\alpha*} = 0$, $x_2^{\alpha*} = 1$, $\lambda_\alpha^* = 1$, $\mu^* = 0$, $\omega^* = 0$ and remaining variables zero, we attain the dual optimum. Observe that in this example the z of the inhomogeneous counterpart to Theorem 1 is $(0, 1)$, so that $z^T Cz = 1$.

1/ Observe that for the inhomogeneous problem, one may no longer further restrict x_α by $Ax_\alpha \leq b$ as in the homogeneous case.

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Unclassified

Security Classification

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Northwestern University	2a. REPORT SECURITY CLASSIFICATION Unclassified
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3. REPORT TITLE

A Generalization of a Theorem of Edmund Eisenberg

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Research paper

5. AUTHOR(S) *(Last name, first name, initial)*

Charnes, Abraham, and Kortanek, Kenneth O.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1965	13	12

8a. CONTRACT OR GRANT NO. Nonr-1228(10)	9a. ORIGINATOR'S REPORT NUMBER(S) Systems Research Memorandum No. 128
b. PROJECT NO. NR 047-021	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* None
d.	

10. AVAILABILITY/LIMITATION NOTICES

Releasable without limitation on dissemination.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Extension and revision of ONR Research Report No. 92	Logistics and Mathematical Statistics Br. Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

The convex duality theory as developed by Charnes-Cooper-Kortanek which is valid for non-differentiable quasi-concave functions is applied to derive a gradient inequality for a non-differentiable function involving a symmetric quadratic form with positivity required only on a given convex polyhedral cone, generalizing a theorem of Edmund Eisenberg. No assumption is needed either that the primal problem attain its extremal value, since the CCK duality theory includes this possibility. The theory directly applies to the non-homogeneous problem where now the given cone is replaced by the intersection of a finite number of half spaces.

Unclassified

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
duality non-differentiable optimizers quasi-concave functions gradient methods semi-infinite programming quadratic programming						

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14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

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Security Classification